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RESEARCH ARTICLE

DISTRIBUTIVITY OF FUZZY ℓ-IDEALS

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ABSTRACT

This paper pursues an investigation on fuzzy ℓ-ideals equipped with lattice ordered groups. We introduced the concept of distributivity in fuzzy ℓ-ideals of lattice ordered group. Also we obtain a characterization theorem for distributive fuzzy ℓ-ideals and some related results are derived.

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1. Introduction

The concept of a fuzzy set was first introduced by Zadeh[17] and this concept was adapted by Goguen [5] to study fuzzy relations. Rosenfeld applied it to group theory and developed the theory of fuzzy groups. Since then researchers in various discipline of Mathematics have been trying to extend their ideas to the broader frame work of the fuzzy setting and the concepts of fuzzy lattice, fuzzy ideal, fuzzy prime ideals in lattice were introduced by many authors. U.M.Swamy and Viswananda Raju [16] developed the theory of fuzzy ℓ-ideals and gave some interesting results. The partially ordered algebraic systems play an important role in algebra. Some important concepts in partially ordered systems are lattice ordered groups and lattice ordered rings. R.Natarajan and J.Vimala [11] introduced the concept of ideals and distributive ℓ-ideals in commutative lattice ordered groups. These concepts play a major role in many branches of Algebra. This provides a sufficient motivation to researchers to review various concepts and results of partially ordered algebraic systems with the frame work of fuzzy setting. G.S.V.SathyaSaiBaba[15] studied fuzzy lattice ordered groups as a mapping from lattice ordered group into a complete lattice. He introduced L-fuzzy ℓ-ideals of fuzzy ℓ-group and developed the theory of fuzzy ℓ-ideals by proving the existence of one to one correspondence between the Lattice of all L-fuzzy ℓ - ideals & the lattice of all L-fuzzy congruence of an l-group G. (Lattice Isomorphism). In this paper we introduce the concept of distributive fuzzy ℓ –ideals and proved that,

(i) Intersection of two distributive fuzzy ℓ-ideals is also distributive.

(ii) Suppose μ is the fuzzy ℓ-ideal of a lattice ordered group G, then the following conditions are equivalent:

- *Fuzzy ℓ-ideal μ is distributive.*
- *The mapping φ: μ_t →[μ_t) is an onto homomorphism where μ_t is a level ℓ-ideal of μ.*
- *The binary relation θ_μ on μ_t is a congruence relation.*

2. Preliminaries

Definition 2.1 A non-empty set G is called a lattice ordered group (ℓ - group)iff

(i) $(G,+)$ is a group

(ii) $(G,\leq)$ is a lattice

(iii) $x \leq y$ implies $a+x+b \leq a+y+b$ for all $a,b,x,y \in G$.

Definition 2.2 A non-empty set G is called a commutative lattice ordered group (ℓ - group) iff

(i) $(G,+)$ is a group

(ii) $(G,\vee,\wedge)$ is a lattice.

(iii) $a+(x\vee y)=(a+x)\vee(a+y)$ and

$a+(x\wedge y)=(a+x)\wedge(a+y)$ for all $a,b,x,y \in G$.

Result 2.3 The above two definitions of ℓ - group are equivalent.

Definition 2.4 Let G be a ℓ - group. A non-empty subset I of G is called an ℓ -ideal of G if

(i) I is a subgroup of G.

(ii) I is a sub lattice of G.

(iii) $0 < x < a$ and $a \in I \Rightarrow x \in 1$

Definition 2.5 A fuzzy set is a pair (X,μ), where X is any non empty set and $\mu : X \to [0,1]$.

Definition 2.6 Let μ be a fuzzy set on a non empty set X and $t \in [0,1]$. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the level set of μ.

Definition 2.7 Let μ be a fuzzy set on a non empty set X. Then the set $\{x \in X / \mu(x) > 0\}$ is called the support of μ and it is denoted by Supp(μ).

3 Fuzzy ℓ -ideals in ℓ -Group

Definition 3.1 Let G be a commutative ℓ -group. A fuzzy set μ of G is said to be fuzzy ℓ -ideal of G if

(i) $\mu(x-y) \geq \mu(x)\wedge\mu(y)$

(ii) $\mu(x\vee y) \geq \mu(x)\wedge\mu(y)$

(iii) $\mu(x\wedge y) \geq \mu(x)\wedge\mu(y)$

(iv) $0 < x < a \Rightarrow \mu(x) \geq \mu(a)$ for all $x,y,a,b \in G$.

Definition 3.2 The union of two fuzzy ℓ -ideals μ_1 and μ_2 of a commutative ℓ -group G denoted by $(\mu_1 \cup \mu_2)$ is a fuzzy subset of G defined by $(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x),\mu_2(x)\}$ for all $x \in G$.

The intersection of two fuzzy ℓ -ideals μ_1 and μ_2 of a commutative ℓ -group G denoted by $(\mu_1 \cap \mu_2)$ is a fuzzy subset of G defined by $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x),\mu_2(x)\}$ for all $x \in G$.

Definition 3.3 Let μ_1 and μ_2 be any two fuzzy ℓ -ideals of a commutative ℓ -group G. Then μ_1 is said to be contained in μ_2 denoted by $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \le \mu_2(x)$ for all $x \in G$. If $\mu_1(x) = \mu_2(x)$ for all $x \in G$ then μ_1 and μ_2 are said to be equal and we can write $\mu_1 = \mu_2$.

Proposition 3.4 Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of a commutative ℓ-group G. If $\mu_1 \subseteq \mu_2$ then $\mu_1 \cup \mu_2 = \mu_2$ and $\mu_1 \cap \mu_2 = \mu_1$.

Proposition 3.5 Let μ_1 and μ_2 be any two fuzzy ℓ -ideals of an commutative ℓ -group G.
Then $\mu_1 \cup \mu_2 \supseteq \mu_1 \cap \mu_2$.

Proposition 3.6 Intersection of any two fuzzy ℓ -ideals of an commutative ℓ -group is a fuzzy ℓ -ideal.

Definition 3.7 If μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G the join of μ_1 and μ_2 defined by

$$(\mu_1 \vee \mu_2)(x) = \sup_{x=y\vee z} \{\min\{\mu_1(y), \mu_2(z)\}\}, \text{ where } x, y, z \in G.$$

The meet of μ_1 and μ_2 defined by

$$(\mu_1 \wedge \mu_2)(x) = \sup_{x=y\wedge z} \{\min\{\mu_1(y), \mu_2(z)\}\}, \text{ where } x, y, z \in G.$$

Proposition 3.8 If μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G then $\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2$.

Proposition 3.9 If μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G then $\mu_1 \wedge \mu_2$ is also a fuzzy ℓ-ideal of G**.**

Proposition 3.10 Let G be a commutative ℓ-group. If μ is a fuzzy ℓ-ideal of G then $Supp(\mu)$ is an ℓ-ideal of G if $Supp \ne \phi$.

Proposition 3.11 If μ_1 is any fuzzy ℓ-ideal of an commutative ℓ-group G then $\mu(1) \le \mu(x) \le \mu(0)$ for all $x \in G$ where 0 is the least element and 1 is the greatest element in G.

Proposition 3.12 Every constant function of a commutative ℓ-group G is a fuzzy ℓ-ideal of G.

Proposition 3.13 *(Characterization Theorem for fuzzy ℓ-ideals)*

Let G be a commutative ℓ - group. A fuzzy set μ of G is a fuzzy ℓ -ideal of G if and only if the set $\mu_t = \{x \in G / \mu(x) \ge t\}$ is an ℓ -ideal of G for all $t \in [0,1]$ with $\mu_t \ne \phi$.

Definition 3.14 Let μ be a fuzzy ℓ-ideal of a commutative ℓ-group G and $t \le \mu(0)$. Then the ℓ-ideal μ_t of G is called a level ℓ-ideal of μ.

4 Distributivity of Fuzzy ℓ-ideals

Definition4.1 Let G be a ℓ-group. A fuzzy ℓ-ideal μ of G is said to be distributive if its level ℓ-ideal in G is distributive.

Proposition 4.*2* *[Characterization theorem for Distributive fuzzy ℓ-ideals]*

Let μ be a fuzzy ℓ -ideal of a ℓ-group G. Then the following are equivalent:

(i) μ is distributive.

(ii) The mapping φ: $\mu_t \rightarrow [\mu_t)$ such that φ(x) = a ∨ x is an onto homomorphism where μ_t is a level ℓ-ideal of μ and $[\mu_t) = \{ x \in \mu_t / x \ge a\}$

(iii)The binary relation θ_μ on μ_t defined by $x \equiv y(\theta_\mu) \leftrightarrow a \vee x = a \vee y$ where $a, x, y \in \mu_t$ is a congruence relation.

Proof

(i)⇨(ii)

Assume that μ is distributive fuzzy ℓ -ideal of G.

⇨The level set μ_t is distributive ℓ -ideal .

We have to prove that $\varphi: \mu_t \rightarrow [\mu_t)$ is an onto homomorphism.

Let $x,y \in \mu_t$ be arbitrary.

Now $\varphi(x \vee y) = a \vee (x \vee y)$

$=(a \vee a) \vee (x \vee y).$

$=(a \vee (a \vee x) \vee y)$

$=((a \vee x) \vee a) \vee Y$

$=(a \vee x) \vee (a \vee y)$

$=\varphi(x) \vee \varphi(y)$

⇨ $\varphi(x \vee y) = \varphi(x) \vee \varphi(y).$

Thus φ preserves ∨.

Now $\varphi(x \wedge y) = a \vee (x \wedge y)$

$= (a \vee x) \wedge (a \vee y)$

$= \varphi(x) \wedge \varphi(y).$

⇨ $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$

Thus φ preserves ∧.

Take any $x \in [\mu_t)$.

⇨ $x \in \mu_t / x \geq a.$

⇨ $x = x \vee a.$

⇨$\varphi(x) = x \vee a = x$

⇨$\varphi(x) = x.$

Hence φ is an onto homomorphism.

(ii)⇨(iii)

Let $x \in \mu_t$ be arbitrary.

⇨ $a \vee x = a \vee x.$

⇨$x \equiv x(\theta_\mu)$

⇨ $\theta_\mu$is reflexive.

Let $x, y \in \mu_t$ be arbitrary.

Suppose $x \equiv y(\theta_\mu)$.

⇨ $a \vee x = a \vee y$

⇨ $a \vee y = a \vee x.$

⇨$y \equiv x(\theta_\mu).$

⇨ θ_μ is symmetric.

Suppose $x \equiv y (\theta_\mu)$, $y \equiv z(\theta_\mu)$.

⇨ $a \vee x = a \vee y$ and $a \vee y = a \vee z$

⇨ $a \vee x = a \vee z.$

⇨$x \equiv z(\theta_\mu).$

⇨ θ_μ is transitive.

Let x,x_1,y,y_1 be arbitrary.

Suppose $x \equiv x_1 (\theta_\mu)$, $y \equiv y_1 (\theta_\mu)$.

⇨ $a \vee x = a \vee x_1$ and $a \vee y = a \vee y_1.$

Now $a \vee (x \vee y) = (a \vee x) \vee y$

$=(a \vee x_1) \vee y$

$=(x_1 \vee a) \vee y$

$=x_1 \vee (a \vee y)$

$= x_1 \vee (a \vee y)$

$= (x_1 \vee a) \vee y)$

$= (a \vee x_1) \vee y_1$

$=a \vee (x_1 \vee y_1).$

Similarly $a \vee (x \wedge y) = a \vee (x_1 \wedge y_1).$

Thus $x \equiv x_1 (\theta_\mu)$, $y \equiv y_1 (\theta_\mu)$. ⇨ $x \vee y \equiv (x_1 \vee y_1) (\theta_\mu)$ and

$x \wedge y \equiv (x_1 \wedge y_1) (\theta_\mu)$.

Hence θ_μ is a congruence relation.

(iii) ⇨ (i)

Let x, y ϵ μ_t be arbitrary.

$a \vee x = (a \vee a) \vee x$

$= a \vee (a \vee x)$

$a \vee y = (a \vee a) \vee y$

$= a \vee (a \vee y)$

⇨ $x \equiv (a \vee x)\ (\theta_\mu)$, $y \equiv (a \vee y)\ (\theta_\mu)$.

⇨ $x \wedge y \equiv (a \vee x) \wedge (a \vee y)\ (\theta_\mu)$.

⇨ $a \vee (x \wedge y) = a \vee [(a \vee x) \Lambda (a \vee y)]$.

⇨ $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$.

⇨ μ_t is distributive ℓ -ideal.

⇨ μ is a distributive fuzzy ℓ -ideal.

Proposition 4.3 Intersection of two distributive fuzzy ℓ -ideals is also distributive.

Proof Let μ_1 , μ_2 be two distributive fuzzy ℓ -ideals.

⇨ The level ℓ -ideals μ_{t1} , μ_{t2} are distributive.

⇨ $\mu_{t1}(x) = \{ x \epsilon G / \mu_1 \geq t_1\}$ and $\mu_{t2}(x) = \{ x \epsilon G / \mu_2 \geq t_2\}$ for $t_1,t_2 \epsilon [0,1]$.

Define $\mu_1 \cap \mu_2 : G \rightarrow [0,1]$ with level ℓ -ideal $(\mu_1 \cap \mu_2)_t = \{ x \epsilon G / \mu_1(x)\ \Lambda\ \mu_2(x) \geq t_1 \vee t_2 \}$.

Clearly $\mu_1 \cap \mu_2$ is a fuzzy ℓ -ideal.

We have to prove that $(\mu_1 \cap \mu_2)_t$ is distributive.

Let x,y,z ϵ $(\mu_1 \cap \mu_2)_t$.

⇨ $\mu_1(x) \wedge \mu_2(x) \geq t_1 \vee t_2$, $\mu_1(y) \wedge \mu_2(y) \geq t_1 \vee t_2$, $\mu_1(z) \wedge \mu_2(z) \geq t_1 \vee t_2$.

⇨ $\mu_1(x) \wedge \mu_2(x) \geq t_1$, $\mu_1(x) \wedge \mu_2(x) \geq t_2$ and $\mu_1(y) \wedge \mu_2(y) \geq t_1$, $\mu_1(y) \wedge \mu_2(y) \geq t_2$

Nd $\mu_1(z) \wedge \mu_2(z) \geq t_1$, $\mu_1(z) \wedge \mu_2(z) \geq t_2$.

⇨ x,y.z ϵ μ_{t1} and x,y.z ϵ μ_{t2}.

⇨ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for x,y.z ϵ $(\mu_1 \cap \mu_2)_t$

⇨ $\mu_1 \cap \mu_2$ is a distributive fuzzy ℓ-ideal.

Proposition 4.4 Let μ_1 , μ_2 be two distributive fuzzy ℓ -ideals. Then $\mu_1\ \Lambda\ \mu_2$ is also distributive.

Proposition 4.5 If μ is a distributive fuzzy ℓ -ideal then Supp(μ) is also distributive.

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